

Mathematical Journal of Okayama University

Volume 38, Issue 1

1996

Article 14

JANUARY 1996

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HOPF MAPS AND TRIALITY

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Introduction: In this note we prove a relation between three kinds of Hopf maps from the seven sphere, using the concept of triality. Let S^7 denote the (unit) sphere in $\mathbb{R}^8 \cong \mathbb{K}$ the Caley number field. Classically, the Hopf projection $h_1 : S^7 \rightarrow S^4$ is expressed in terms of quaternions.

In §1 we give an expression for h_1 in terms of Cayley numbers, which seems to be quite natural in a sense described in the text.

In §2 we employ the concept of triality [1], to show that a certain relation holds between three kinds of Hopf-type maps from S^7 .

In §3 we use the principal S^3 -bundles over S^7 and the geometry of the exceptional Lie group G_2 to determine a map that generates $\pi_7 Spin(5)$, though not with an explicit formula.

In §4 we use E. Cartan's inclusion of a symmetric space G/H , quotient of the symmetric pair $H \subset G$, into G as a totally geodesic submanifold and the result of §2 to give a characterization of the space $G_2/SO(4)$ as a totally geodesic submanifold of G_2 .

We indicate how the considerations above could lead to an explicit description of a map generating $\pi_7 Sp(2) \cong \mathbb{Z}$ [10].

Acknowledgement: The authors are indebted to Professor Juno Mukai for helpful comments and encouragement.

§1. The Hopf map.

Let S^7 denote the unitary Cayley numbers $\mathbb{K} \cong \mathbb{R}^8$ and denote its elements by Greek letters α, β, \dots . We begin by expressing the Hopf projection $S^3 \times S^7 \xrightarrow{h_1} S^4$ in the context of Cayley multiplication. Recall [6] that this multiplication is defined on pairs of quaternions $\alpha = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\beta = \begin{pmatrix} c \\ d \end{pmatrix}$ as

$$\alpha\beta = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - \bar{d}b \\ da + b\bar{c} \end{pmatrix}.$$

If α is unitary, i.e., $|a|^2 + |b|^2 = 1$, the projection h_1 is classically considered as the quotient map of the free $Sp(1) \cong S^3$ action on S^7 , say

from the right by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} aq \\ bq \end{pmatrix}.$$

An invariant projection on $\mathcal{Q}P^1 \cong S^4$ is usually taken to be $h_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} |a|^2 - |b|^2 \\ 2a\bar{b} \end{pmatrix}$, where $2a\bar{b}$ lives in D^4 , the unit disk of $\mathbb{R}^4 \cong \mathcal{Q}$, the quaternions, while the first coordinate $|a|^2 - |b|^2$ is in $[-1, 1] \subset \mathbb{R}$ and can be thought of as representing the “necessary height” for the pair to be in S^4 , i.e., a unitary vector in \mathbb{R}^5 .

Given any element (J, K) of $V_{7,2}$, the orthonormal 2-frames in $\mathbb{R}^7 \cong Im\mathbb{K}$ with the euclidean metric, observe that it defines an inclusion of \mathcal{Q} in \mathbb{K} by the correspondence of the usual units $i \mapsto J$, $j \mapsto K$ and $k \mapsto JK$.

The elements J and K can be thought of as belonging to the purely imaginary equator S^6 of S^7 , as they satisfy $J^2 = K^2 = -1$ and $JK = -KJ$, being orthogonal to each other.

Lemma 1: The map $\delta : S^7 \rightarrow S^7$ defined by $\delta(\alpha) = (J\bar{\alpha})(\alpha K)$ satisfies $\langle \delta(\alpha), 1 \rangle = 0$, $\langle \delta(\alpha), J \rangle = 0$ and $\langle \delta(\alpha), K \rangle = 0$.

Proof. $\langle (J\bar{\alpha})(\alpha K), 1 \rangle = -\langle \alpha J, \alpha K \rangle = -\langle J, K \rangle = 0$ by the invariance of the euclidean metric with respect to Cayley multiplication. Similarly, $\langle (J\bar{\alpha})(\alpha K), J \rangle = -\langle (\alpha J)J, \alpha K \rangle = \langle \alpha, \alpha K \rangle = \langle 1, K \rangle = 0$ and analogously for K in place of J . QED

Theorem 1: (i) The image of δ lies in the unitary four sphere in the 5-dimensional linear subspace of \mathbb{K} perpendicular to $1, J$ and K .

(ii) The map $h : S^7 \rightarrow S^4$ defined by

$$(h) \quad h(\alpha) = (e_1 \bar{\alpha})(\alpha e_2)$$

is in a way to be explained in the Remarks following the proof, essentially the Hopf map, where e_1 and e_2 are $\begin{pmatrix} i \\ 0 \end{pmatrix}$ and $\begin{pmatrix} j \\ 0 \end{pmatrix}$ in $\mathcal{Q} \oplus \mathcal{Q} \cong \mathbb{K}$.

Proof: (i) is immediate. To show (ii) observe that $\bar{\alpha} = \begin{pmatrix} \bar{a} \\ -b \end{pmatrix}$ and

therefore

$$\begin{aligned} h(\alpha) &= \left[\begin{pmatrix} i \\ 0 \end{pmatrix} \begin{pmatrix} \bar{a} \\ -b \end{pmatrix} \right] \left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} j \\ 0 \end{pmatrix} \right] = \begin{pmatrix} i\bar{a} \\ -bi \end{pmatrix} \begin{pmatrix} aj \\ -bj \end{pmatrix} \\ &= \begin{pmatrix} i\bar{a}aj + j\bar{b}bi \\ -bj i\bar{a} + bi j\bar{a} \end{pmatrix} = \begin{pmatrix} (|a|^2 - |b|^2)k \\ 2bk\bar{a} \end{pmatrix}, \end{aligned}$$

where $k = ij$ in \mathcal{Q} .

A homotopy between h and h_1 is constructed as follows: $h \begin{pmatrix} a \\ b \end{pmatrix}$ is homotopic to $\begin{pmatrix} |a|^2 - |b|^2 \\ 2b\bar{a} \end{pmatrix}$ through $\begin{pmatrix} a \\ b \end{pmatrix}, t \mapsto \begin{pmatrix} |a|^2 - |b|^2 \\ 2b(\cos(t) + \sin(t)k)\bar{a} \end{pmatrix}$, $0 \leq t \leq \pi/2$, and $\begin{pmatrix} |a|^2 - |b|^2 \\ 2b\bar{a} \end{pmatrix}$ is homotopic to $\begin{pmatrix} |b|^2 - |a|^2 \\ -2a\bar{b} \end{pmatrix} = -h_1 \begin{pmatrix} a \\ b \end{pmatrix}$ by observing that these two maps just differ by a change of sign in the first two coordinates. Therefore h is homotopic to $-h_1$. Since both h and h_1 are essentially the same Hopf map, modulo choice of orientation, we have the claimed result. QED

Remarks: i) The map $-h_1$ is $(-\iota_4) \circ h_1$, where $-\iota_4$ is the antipodal map in S^4 . Its homotopy class in $\pi_7(S^4)$ is $[h_1] \pm \sum w$ where $[h_1]$ is the class of h_1 and $\sum w$ is the suspension of the Blakers-Massey element that generates $\pi_6 S^3$. The ambiguity of the sign depends on the choice of orientation.

ii) The classification of S^3 -principal bundles over S^4 , by $\pi_4 BS^3 \cong \pi_3 S^3 \cong \mathbb{Z}$, implies, through the exact homotopy sequence of such a fibration, that there are precisely two total spaces whose third homotopy group is zero and are, therefore, homeomorphic to S^7 .

These two bundles correspond to 1 and -1 in \mathbb{Z} and are represented by h_1 and $-h_1$ in an order that depends on the choice of orientation.

iii) From the homotopy ladder of the pullback diagram

$$\begin{array}{ccc} S^3 & & S^3 \\ \vdots & & \vdots \\ P_n & \longrightarrow & ES^3 \\ \downarrow & & \downarrow \\ S^4 & \xrightarrow{\quad n \quad} & BS^3, \end{array}$$

where n denotes a map of degree n in the 4th homotopy group, it follows

that $\pi_3(P_n) \cong \mathbb{Z}_n$.

iv) The map h is the invariant projection of the following free $Sp(1)$ action on S^7 :

$$\begin{pmatrix} a \\ b \end{pmatrix} q = \begin{pmatrix} aq \\ bkq\bar{k} \end{pmatrix} = \begin{pmatrix} aq \\ -bkqk \end{pmatrix}.$$

v) The expression (h) for the Hopf map emphasizes that it reflects the non-associativity of the Cayley product. This is analogous to the fact that the Hopf map $h_0 : S^3 \rightarrow S^2$ defined by $h_0(q) = qi\bar{q}$, reflects the non commutativity of the quaternionic product.

vi) The expression for h in (h) can be seen as part of the classical expression for the "next" Hopf map

$$h_2 : S^{15} \rightarrow S^8, \quad \text{as follows:}$$

Let $\begin{pmatrix} A \\ B \end{pmatrix}$ be a pair of Cayley numbers with

$$|A|^2 + |B|^2 = 1, \quad \text{i.e.,} \quad \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{in} \quad S^{15} \subseteq \mathbb{R}^{16} \cong \mathbb{K} \oplus \mathbb{K}.$$

An algebraic expression for h_2 is again

$$h_2 \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} |A|^2 - |B|^2 \\ 2\bar{A}B \end{pmatrix}.$$

Consider now the inclusion of S^7 in S^{15} by $\alpha \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} -\alpha e_1 \\ \alpha e_2 \end{pmatrix}$ and compose with h_2 .

§2. Other Hopf maps

Consider the following Hopf-type maps [11],

$$\begin{aligned} H' : S^7 \times S^6 &\rightarrow S^6 \quad \text{defined by} \\ H'(\alpha, J) &= \alpha J \bar{\alpha}. \end{aligned}$$

Observe that for a fixed J , for example $J = e_1$, the map $\alpha \mapsto \alpha e_1 \bar{\alpha}$ generates $\pi_7(S^6) \cong \mathbb{Z}_2$.

Similarly, the map $H : S^7 \rightarrow V_{7,2}$ with $H(\alpha) = \{\alpha e_1 \bar{\alpha}, \alpha e_2 \bar{\alpha}\}$ generates $\pi_7(V_{7,2}) \cong \mathbb{Z}_4$. This follows easily from [11] and the exact homotopy

sequence of the fibrations $SO(5) \cdots SO(7) \longrightarrow V_{7,2}$ and $S^5 \cdots V_{7,2} \longrightarrow S^6$.

In order to relate the above described Hopf maps we consider the bundle

$$S^4 \cdots V_{7,3} \xrightarrow{p} V_{7,2}$$

with the obvious projection $p(J, K, L) = (J, K)$. Observe that the map φ of Lemma 1 furnishes a family of sections of p parametrized by S^7 , through the Hopf map:

$$x : S^7 \times V_{7,2} \rightarrow V_{7,3}, \quad \text{by} \quad x(\alpha, (J, K)) = (J, K, (J\bar{\alpha})(\alpha K)).$$

For any fixed (J, K) in $V_{7,2}$ the map $\alpha \longmapsto (J\bar{\alpha})(\alpha K)$ is the Hopf map from S^7 to the fiber $p^{-1}(J, K) \cong S^4$. Note that the Cayley multiplication also furnishes a section χ of p as follows:

$$\chi(J, K) = (J, K, JK),$$

where JK is the Cayley product of J and K .

Conjugation by elements of S^7 preserves $V_{7,2}$, i.e., if α is in S^7 then $(\alpha J\bar{\alpha}, \alpha K\bar{\alpha})$ is in $V_{7,2}$ for (J, K) in $V_{7,2}$. The non-associativity of Cayley numbers prevents us from calling this an action.

The concept of triality [1], [11] provides a relation between the Hopf maps defined above.

Define $h : S^7 \times V_{7,2} \rightarrow S^6$ by $h(\alpha, (J, K)) = (J\bar{\alpha})(\alpha K)$.

Proposition 1: For any α in S^7 , m, n in \mathbb{Z} and for $\star = (e_1, e_2)$, the base point of $V_{7,2}$, the following relation holds

$$h(\alpha^m, H(\alpha^n)) = H'(\alpha^n, h(\alpha^{m+3n}, \star))$$

where $h(\beta, \star)$ is defined to be $(e_1\bar{\beta})(\beta e_2)$.

Proof: Recall that the principle of triality assigns to each A in $SO(8)$ a pair (B, C) in $SO(8) \times SO(8)$, modulo common sign, such that

$$A(xy) = B(x)C(y) \quad \text{for all } x, y \text{ in } \mathbb{K},$$

both products being Cayley multiplications.

It was shown in [11] that if A is the conjugation by a unit Cayley number α in S^7 , then $B(x) = \alpha x \alpha^2$ and $C(y) = \bar{\alpha}^2 y \bar{\alpha}$, i.e., $\alpha(xy)\bar{\alpha} = (\alpha x \alpha^2)(\bar{\alpha}^2 y \bar{\alpha})$.

Let $x = \bar{\alpha}\xi\alpha$ and $y = \bar{\alpha}\eta\alpha$, so we have

$$\alpha[(\bar{\alpha}\xi\alpha)(\bar{\alpha}\eta\alpha)]\bar{\alpha} = (\alpha\bar{\alpha}\xi\alpha\alpha^2)(\bar{\alpha}^2\bar{\alpha}\eta\alpha\bar{\alpha}) = (\xi\alpha^3)(\bar{\alpha}^3\eta),$$

since any subalgebra of \mathcal{K} generated by two elements is associative. Therefore, we also have

$$\bar{\alpha}[(\alpha x\bar{\alpha})(\alpha y\bar{\alpha})]\alpha = (x\bar{\alpha}^3)(\alpha^3 y),$$

or

$$(\mu) \quad (\alpha x\bar{\alpha})(\alpha y\bar{\alpha}) = \alpha[(x\bar{\alpha}^3)(\alpha^3 y)]\bar{\alpha}$$

for all α in S^7 , x, y in \mathcal{K} [8].

With the above notation we have now,

$$\begin{aligned} h(\alpha^m, H(\alpha^n)) &= h(\alpha^m, (\alpha^n e_1 \bar{\alpha}^n, \alpha^n e_2 \bar{\alpha}^n)) = \\ &= [(\alpha^n e_1 \bar{\alpha}^n) \bar{\alpha}^m][\alpha^m (\alpha^n e_2 \bar{\alpha}^n)] = \\ &= [(\alpha^n (e_1 \bar{\alpha}^m) \bar{\alpha}^n) \bar{\alpha}^n][\alpha^n (\alpha^m e_2) \bar{\alpha}^n] \quad (\text{by } (\mu)) \\ &= \alpha^n [(e_1 \bar{\alpha}^{m+3n})(\alpha^{m+3n} e_2)] \bar{\alpha}^n = \\ &= \alpha^n h(\alpha^{m+3n}, \star) \bar{\alpha}^n = H'(\alpha^n, h(\alpha^{m+3n}, \star)). \end{aligned}$$

Example: $(\alpha e_1 \bar{\alpha})(\alpha e_2 \bar{\alpha}) = \alpha h(\alpha^3) \bar{\alpha}$.

§3. S^3 -Principal Bundles over S^7

One could use the formula of the Example above to look for an explicitly defined section of the (trivial) S^3 -principal bundle $E(12)$ over S^7 [5] or equivalently, for an explicitly written generator of $\pi_7 Sp(2) \equiv \mathbb{Z}$ ([10, p. 238]), in a sense that we describe below. We denote by $E(n)$ the pull-back of the bundle $S^3 \dots Sp(2) \rightarrow S^7$ by a degree n self map f_n of S^7 . For example, $f_n(\alpha) = \alpha^n$.

Consider the generator $Sp(2)$ of the S^3 -principal bundles over S^7 as the pull back of $-h$ by h over S^7 as follows:

$$Sp(2) = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{with} \quad AA^* = A^*A = I \right\}$$

where a, b, c and d are quaternions

$$\begin{array}{ccc} Sp(2) & \xrightarrow{2^{nd} col.} & S^7 \\ 1^{st} col. \downarrow & & \downarrow -h \\ S^7 & \xrightarrow{h} & S^4 \end{array}$$

The diagram above is commutative by the definition of $Sp(2)$, i.e., because $|a|^2 - |b|^2 = -(|c|^2 - |d|^2)$ and $2a\bar{b} = -2c\bar{d}$. As $E(k)$ is the pull-back $f_k^*(Sp(2)) = f_k^*(E_1)$, where $f_k : S^7 \rightarrow S^7$ is a map of degree k [5] for $k = 2, 3, \dots$, it follows that to write down a section of the trivial $E(12)$ is equivalent to producing a map

$$\beta : S^7 \longrightarrow S^7 \quad \text{with} \quad -h \circ \beta(\alpha) = h(\alpha^{12})$$

and the matrix $(\alpha^{12}, \beta(\alpha))$ of $Sp(2)$ represents a generator of $\pi_7 Sp(2)$.

From the relevant part of the exact homotopy sequence of the principal fibration $S^3 \dots G_2 \rightarrow V_{7,2}$ one can readily observe that the 2-primary component of $\pi_6(S^3) \cong \mathbb{Z}_{12} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4$ is related to $\pi_7 V_{7,2} \cong \mathbb{Z}_4$. The 3-primary component is related to $\pi_6(G_2) \cong \mathbb{Z}_3$ ([7], [3]).

A strategy for obtaining a section of $E(12)$ is the following:

The principal S^3 -bundle $S^3 \dots G_2 \xrightarrow{p} V_{7,2}$, where G_2 is the exceptional Lie group of automorphisms of \mathbb{K} , implies that there exists a lifting to G_2 of the (homotopically trivial) map $\tau : S^7 \longrightarrow V_{7,2}$ defined by $\tau(\alpha) = (\alpha^4 e_1 \bar{\alpha}^4, \alpha^4 e_2 \bar{\alpha}^4)$.

Recall now ([13]) that the columns of all elements of $G_2 \subseteq SO(7)$ satisfy the same relation as the purely imaginary units of \mathbb{K} , i.e., $3^{rd} col. = (1^{st} col.)(2^{nd} col.)$, $5^{th} col. = (1^{st} col.)(4^{th} col.)$, $6^{th} col. = (2^{nd} col.)(4^{th} col.)$, $7^{th} col. = (3^{rd} col.)(4^{th} col.)$. As the projection p above is precisely on the first two columns, it follows that to lift τ one should just determine a fourth column, i.e., an element of S^7 that is perpendicular, in the euclidean metric of \mathbb{R}^8 , to $\alpha^4 e_1 \bar{\alpha}^4, \alpha^4 e_2 \bar{\alpha}^4$ as well as to their Cayley product that represents the third column and is according to the **Example** above equal $\alpha^4 h(\alpha^{12}) \bar{\alpha}^4$. As conjugation by a unitary element is \mathbb{R}^8 , in our case by α^4 , is an isometry of the euclidean metric, one is looking for a map $\varphi : S^7 \longrightarrow S^4$, such that $\langle \varphi(\alpha), h(\alpha^{12}) \rangle = 0$, for all α is S^7 . Such a φ exists, from the above discussion. If $C(\bar{\alpha}^4)$ represents conjugation by $\bar{\alpha}^4$ in $SO(7)$ and $\psi : S^7 \longrightarrow G_2$ is a lifting of τ , according to the above, then $C(\bar{\alpha}^4) \circ \psi(\alpha)$ is a matrix in $SO(5)$ and we have

Proposition 2: $C(\bar{\alpha}^4) \circ \psi(\alpha)$ generates $\pi_7 SO(5)$ and $\pi_7 Spin(5)$ in the sense of triality.

Proof: Observe that $C(\bar{\alpha}^4) \circ \psi(\alpha)$ is a 7×7 matrix with the following columns: $e_1, e_2, h(\alpha^{12}), \varphi(\alpha), \bar{\alpha}^4[(\alpha^4 e_1 \bar{\alpha}^4)(\alpha^4 \varphi(\alpha) \bar{\alpha}^4)]\alpha^4, \bar{\alpha}^4[(\alpha^4 e_2 \bar{\alpha}^4)(\alpha^4 \varphi(\alpha) \bar{\alpha}^4)]\alpha^4$ and $\bar{\alpha}^4[(\alpha^4 h(\alpha^{12}) \bar{\alpha}^4)(\alpha^4 \varphi(\alpha) \bar{\alpha}^4)]\alpha^4$. Call this matrix $A(\alpha)$ and we have the element $(A(\alpha), B(\alpha), C(\alpha))$ in $Spin(5) \subseteq Spin(8)$ according to triality, see also [12]. We know that the projection $Spin(5) \rightarrow S^7$ corresponding to the first column projection $Sp(2) \rightarrow S^7$ used above, corresponds to $B(\alpha)(1)$. If the degree of $\alpha \mapsto B(\alpha)(1)$ as a map from S^7 to itself is ± 12 , then the map $\alpha \mapsto (A(\alpha), B(\alpha), C(\alpha))$ represents a generator of $\pi_7 Spin(5) = \pi_7 Sp(2)$ and $\alpha \mapsto A(\alpha)$ is a generator of $\pi_7 SO(5)$, as follows from the exact homotopy sequence of $S^3 \cdots Spin(5) \rightarrow S^7$ and the identification $Spin(5) \cong Sp(2)$.

Let L_β , resp. R_β , denote left, resp. right, Cayley multiplication by β , then using the triality, we have

$$C(\bar{\alpha}^4) \equiv (L_{\bar{\alpha}^4} \circ R_{\alpha^4}, L_{\bar{\alpha}^4} \circ R_{\bar{\alpha}^8}, L_{\alpha^8} \circ R_{\alpha^4}) \quad \text{in } Spin(7)$$

and

$$\psi(\alpha) \equiv (\psi(\alpha), \psi(\alpha), \psi(\alpha)) \quad \text{in } G_2 \subseteq Spin(7).$$

Recall that G_2 is characterized by its elements being of the form (A, A, A) in $Spin(7)$, by its definition as the automorphism group of \mathbb{K} . Therefore

$$C(\bar{\alpha}^4) \circ \psi(\alpha) = (L_{\bar{\alpha}^4} \circ R_{\alpha^4} \circ \psi(\alpha), L_{\bar{\alpha}^4} \circ R_{\bar{\alpha}^8} \circ \psi(\alpha), L_{\alpha^8} \circ R_{\alpha^4} \circ \psi(\alpha))$$

and $B(\alpha)(1) = L_{\bar{\alpha}^4} \circ R_{\bar{\alpha}^8} \circ \psi(\alpha)(1) = \bar{\alpha}^4[\psi(\alpha)(1)]\bar{\alpha}^8 = \bar{\alpha}^{12}$, since $\psi(\alpha)(1) = 1$. Therefore $C(\bar{\alpha}^4) \circ \psi(\alpha)$ generates $\pi_7 Spin(5)$ in the manner described above. QED

Suppose now that we are given φ (and ψ) and we want to retrieve β , a section of $E(12)$. By the above we can construct $g_1 \equiv g_1(\varphi)$ a generator of $\pi_7 Spin(5)$. By an easy argument [4], it follows that the corresponding map $g : S^7 \rightarrow Sp(2)$ has columns α^{12} and $B(\alpha)(e_4)$. From the diagram

below

$$\begin{array}{ccccc}
 S^7 & \xrightarrow{g} & Sp(2) & \xrightarrow{2^{nd} col.} & S^7 \\
 & \searrow \lambda & \downarrow 1^{st} col. & & \downarrow -h \\
 & & S^7 & \xrightarrow{h} & S^4
 \end{array}$$

we have $\lambda = 1^{st} col. g$, $deg(\lambda) = 12$, and $h(\lambda(\alpha)) = -h(2^{nd} col. g(\alpha)) = -h(B(\alpha)(e_4))$.

In other words, the map $S^7 \rightarrow S^7 \times Sp(2)$ with $\alpha \mapsto (\alpha, g(\alpha))$ is a section of $E(12)$.

Observe that the second column of the matrix $g(\alpha)$ in $Sp(2)$ has degree -12 too, as a map from S^7 to itself: As we saw $2^{nd} col. g(\alpha) = B(\alpha)(e_4) = \bar{\alpha}^4(\psi(\alpha)(e_4))\bar{\alpha}^8 = \bar{\alpha}^4(\alpha^4\varphi(\alpha)\bar{\alpha}^4)\bar{\alpha}^8 = \varphi(\alpha)\bar{\alpha}^{12}$, but $\varphi(\alpha)$ has degree zero since it lands in S^4 .

§4. E. Cartan's method

To construct a natural (though, fatally nullhomotopic) map from S^7 to G_2 we employ Elie Cartan's method of embedding symmetric spaces into Lie groups as totally geodesic submanifolds [2, p. 77].

In our case, we begin with the generator of $\pi_7(V_{7,2})$ described earlier, i.e., $(\alpha e_1 \bar{\alpha}, \alpha e_2 \bar{\alpha})$ and apply to it the section χ to obtain the element $(\alpha e_1 \bar{\alpha}, \alpha e_2 \bar{\alpha}, (\alpha e_1 \bar{\alpha})(\alpha e_2 \bar{\alpha})) = (\alpha e_1 \bar{\alpha}, \alpha e_2 \bar{\alpha}, \alpha h(\alpha^3) \bar{\alpha})$ in $V_{7,3}$. Cartan's method does not apply directly in this case, since $(SO(7), SO(4))$ is not a symmetric pair ($(SO(7), SO(3) \times SO(4))$ is one). We can, however, consider the conjugate orbit of the matrix $A = \begin{pmatrix} I_3 & 0 \\ 0 & -I_4 \end{pmatrix}$, which

amounts to $V_{7,2} \rightarrow G_2/SO(4) \xrightarrow{\text{Cartan}} G_2$, since A is in $G_2 \subseteq SO(7)$.

We will employ the Moufang identities to prove

Theorem 2: The Cartan inclusion Λ of $G_2/SO(4)$ in G_2 is $\Lambda([B]) = L_{b_3} \circ L_{b_2} \circ L_{b_1}$, where b_i , $i = 1, 2, 3$ are the first three columns of any matrix in the class $[B]$ in $G_2/SO(4)$.

Proof: Note that A is the composition of four reflections in $\mathbb{R}^7 = Im \mathbb{K}$, each one with respect to the hyperplane perpendicular to e_4, e_5, e_6 and e_7 . If v is in S^6 , unitary vector in \mathbb{R}^7 , the reflection in the hyperplane perpendicular to v , denoted by R_v is $R_v(x) = vxv$. Therefore,

$$A(x) = e_7(e_6(e_5(e_4 x e_4)e_5)e_6)e_7.$$

for the same reason $-A = \begin{pmatrix} -I_3 & 0 \\ 0 & I_4 \end{pmatrix}$, that belongs to the “negative” connected component of $O(7)$, is equal to

$$-A(x) = e_3(e_2(e_1xe_1)e_2)e_3 \quad , \quad \text{for all } x \in \mathbb{R}^7,$$

and $A(x) = -e_3(e_2(e_1xe_1)e_2)e_3$.

Employing the following Mounfang identity [11], [8]

$$a(xy)a = (ax)(ya)$$

we obtain

$$\begin{aligned} A(x) &= -e_3\{e_2[(e_1x)e_1]e_2\}e_3 \\ &= -e_3\{e_2[(e_1x)][e_1e_2]\}e_3 = \\ &= -e_3\{e_2[(e_1x)]e_3\}e_3 \\ &= -e_3\{e_2[(e_1x)]\}e_3^2 = e_3(e_2(e_1x)). \end{aligned}$$

Recall now that the columns of $[B]$ satisfy $b_3 = b_1b_2$, so that the matrix B can be choosen to belong to G_2 , as the rest of the columns beyond the third don’t matter. In this case we can assume that B distributes over Cayley products and we have:

$$\begin{aligned} BAB^{-1}(x) &= BA(B^{-1}(x)) = B[e_3(e_2(e_1B^{-1}(x)))] = \\ &= b_3(b_2(b_1x)) = L_{b_3} \circ L_{b_2} \circ L_{b_1}(x) \quad \text{as claimed.} \end{aligned} \quad \text{QED}$$

Corollary 1: The Cartan inclusion of $G_2/SO(4)$ in G_2 is represented by: (J, K) in $V_{7,2}$ goes to $L_{JK} \circ L_K \circ L_J$ in G_2 , using the projection s of the fibration

$$SO(3) \dots V_{7,2} \xrightarrow{s} G_2/SO(4).$$

Let now Ψ be the composition $\Lambda \circ s \circ H$, where H is the generator of $\pi_7(V_{7,2})$ from §2, s is the projection from $V_{7,2}$ to $G_2/SO(4)$ and Λ is the Cartan inclusion of the symmetric space $G_2/SO(4)$ into G_2 described above.

Corollary 2: The resulting map Ψ from S^7 to G_2 has columns $\Psi(\alpha)(e_i) = [\alpha h(\alpha^3)\bar{\alpha}]\{(\alpha e_2\bar{\alpha})[(\alpha e_1\bar{\alpha})e_i]\}$ for $i = 1, 2, \dots, 7$.

This map is the lifting to G_2 of the homotopically trivial map $\mu : S^7 \longrightarrow V_{7,2}$, with $\mu = p \circ \Psi$. In order to construct a generator of $\pi_7Sp(2)$

one should produce a map $\Phi : S^7 \rightarrow SO(7)$ with $[\Phi] = \pm 4$ in $\pi_7 SO(7)$, such that the first two columns of $\Phi(\alpha)$ are precisely $\mu(\alpha)$. Note that $\alpha h(\alpha^3)\bar{\alpha}$ is just $(\alpha e_1 \bar{\alpha})(\alpha e_2 \bar{\alpha})$ and that each $\alpha e_i \bar{\alpha}$ is a representative of the generator of $\pi_7(S^6)$.

Problem: (i) We do not know if there exists a reasonable formula describing such a Φ .

(ii) Similarly, no explicit formula for a map $\varphi : S^7 \rightarrow S^4$, described above, that would lead to a lifting ψ of τ , is known to the authors either.

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(Received August 26, 1996)